Analysis and Design of Analog Integrated Circuits
Lecture 13

Basics of Noise

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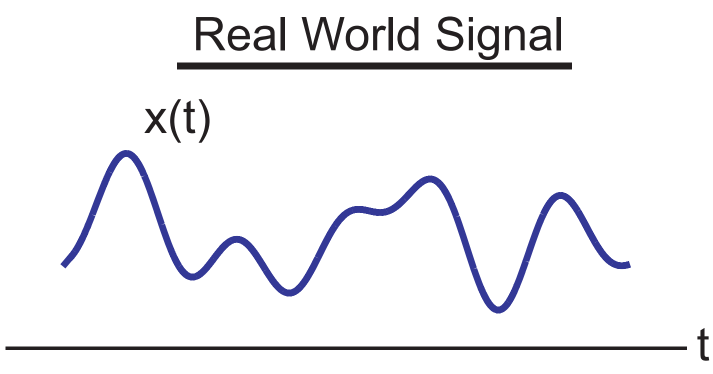
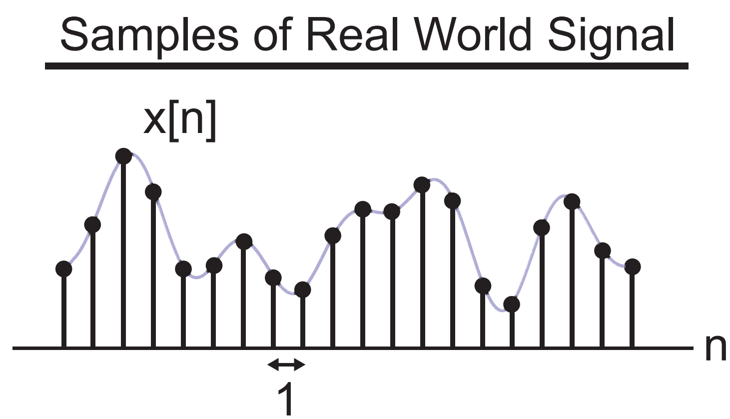
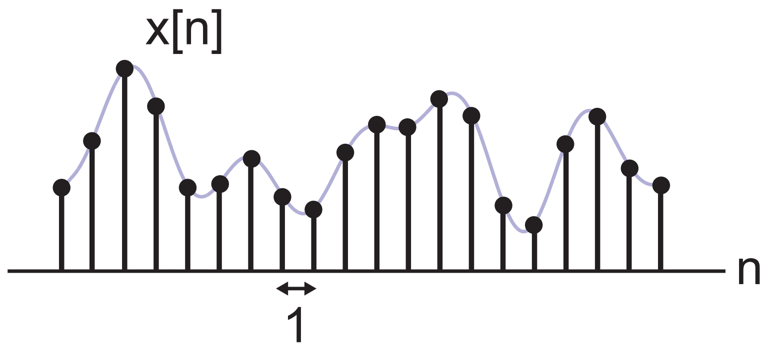
Continuous-Time Versus Discrete-Time Signals





- **Real world signals, such as acoustic signals from speakers and RF signals from cell phones, are *continuous-time* in nature**
- **Digital processing of signals requires samples of real world signals, which yields *discrete-time* signals**
- **Analog circuits are used to sample and digitize real world signals for use by digital processors**
- **It is useful to study discrete-time signals when examining the issue of noise**
  - **Many insights can be applied back to continuous-time signals**

Definition of Mean, Power, and Energy



- **DC average or mean, μ_x, is defined as**

$$\mu_x = \frac{1}{N}\sum_{k=0}^{N-1} x[n]$$

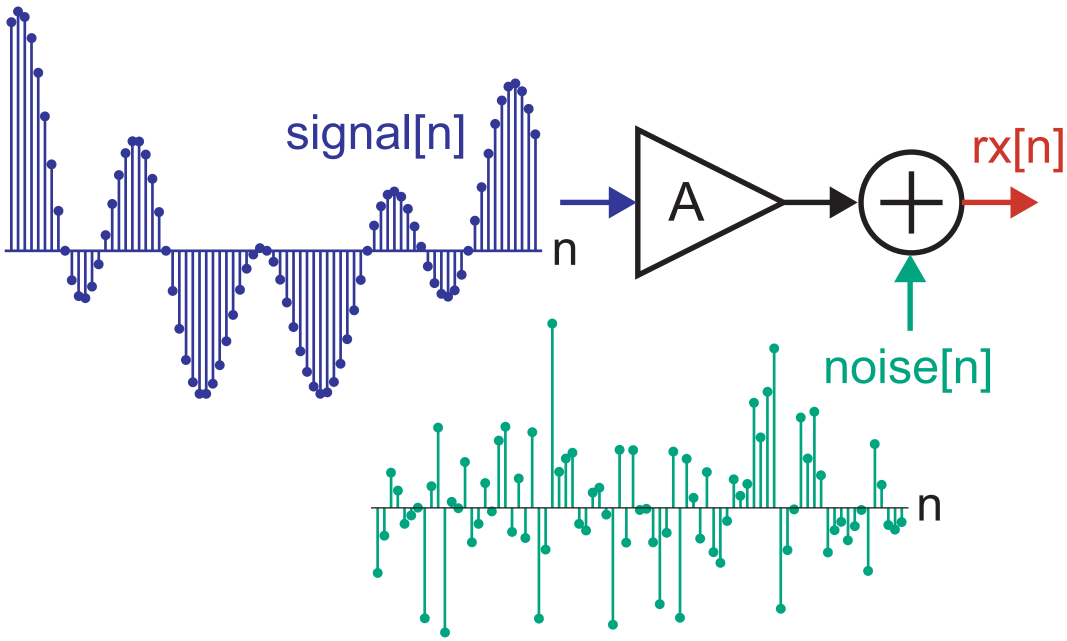
- **Power, P_x, and energy, E_x, are defined as**

$$P_x = \frac{1}{N}\sum_{k=0}^{N-1} x[n]^2 \qquad E_x = \sum_{k=0}^{N-1} x[n]^2$$

  - **For many systems, we often remove the mean since it is often irrelevant in terms of *information*:**

$$\tilde{P}_x = \frac{1}{N}\sum_{k=0}^{N-1} (x[n]-\mu_x)^2 \qquad \tilde{E}_x = \sum_{k=0}^{N-1} (x[n]-\mu_x)^2$$

Definition of Signal-to-Noise Ratio



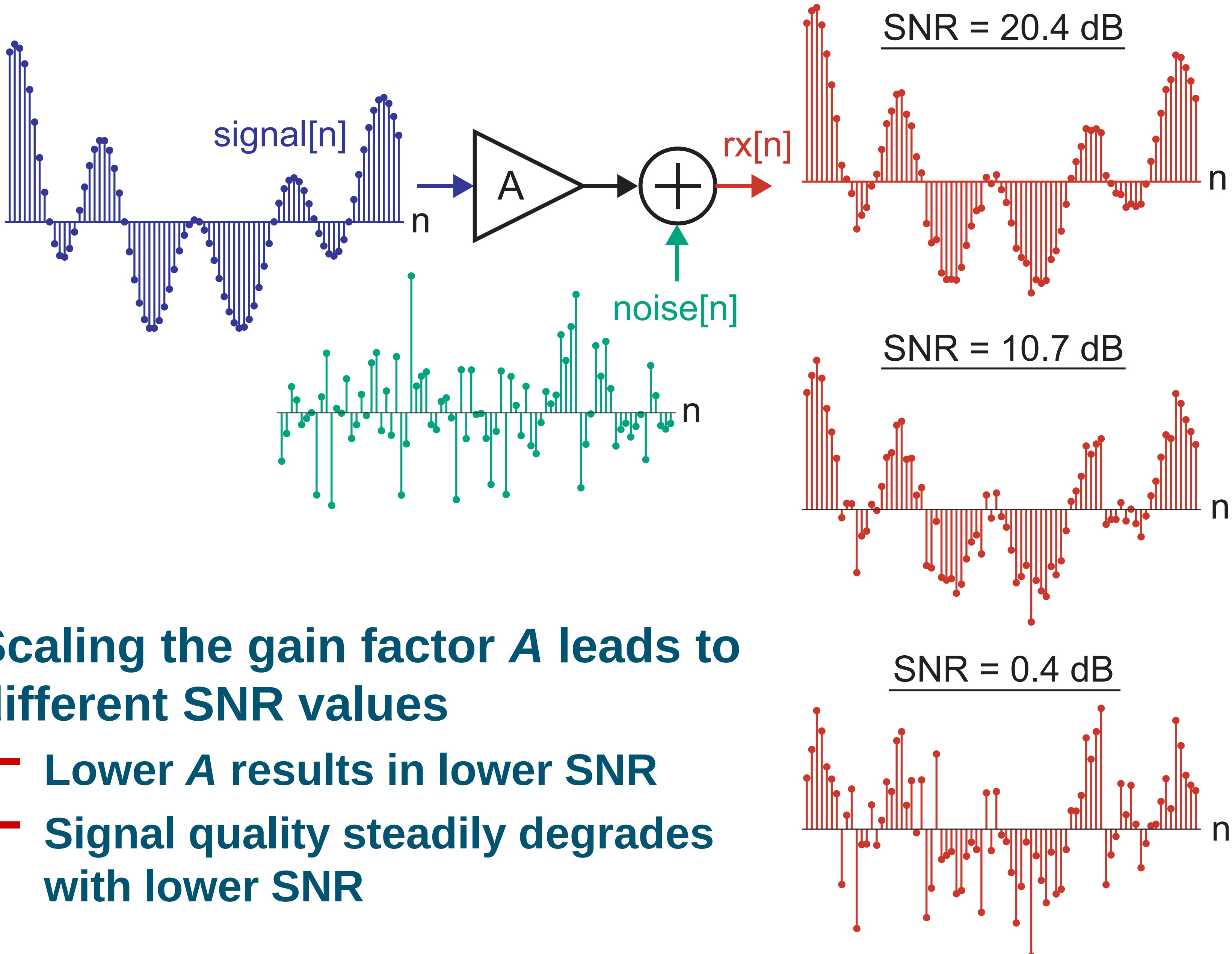
- **Signal-to-Noise ratio (SNR) indicates the relative impact of noise on system performance**

$$\mathrm{SNR} = \frac{\tilde{P}_{signal}}{\tilde{P}_{noise}}$$

- **We often like to use units of *dB* to express SNR:**

$$\mathrm{SNR\ (dB)} = 10\log\left(\frac{\tilde{P}_{signal}}{\tilde{P}_{noise}}\right)$$

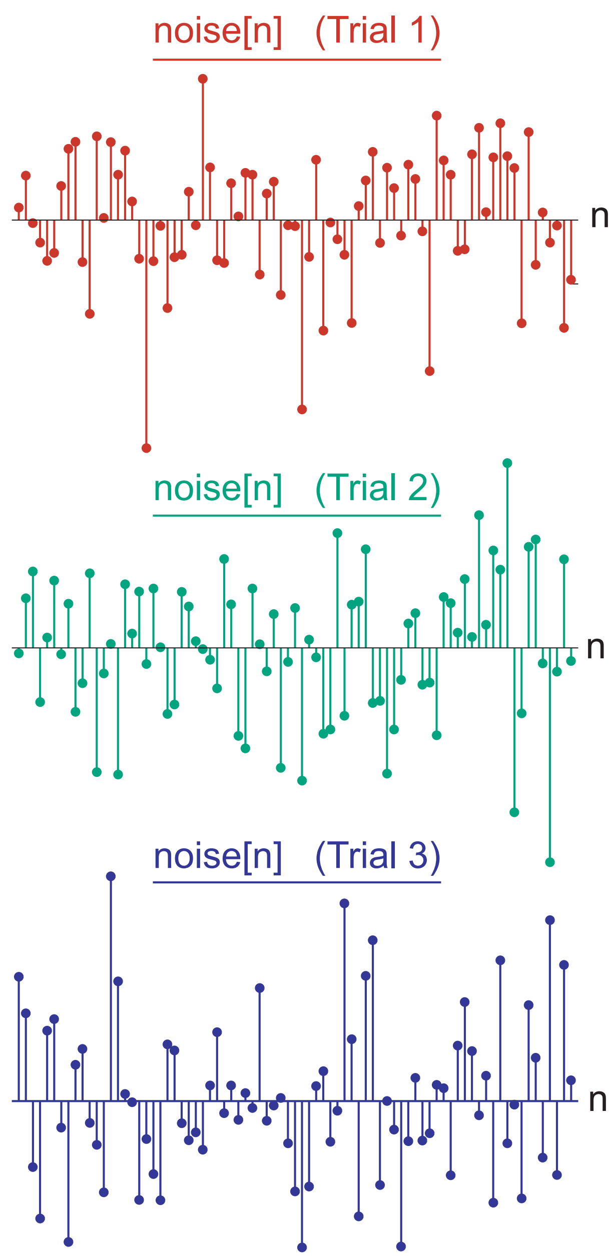
SNR Example



- **Scaling the gain factor *A* leads to different SNR values**
  - **Lower *A* results in lower SNR**
  - **Signal quality steadily degrades with lower SNR**

Analysis of Random Processes

- **Random processes, such as noise, take on different sequences for different trials**
  - **Think of trials as different measurement intervals from the same experimental setup**
- **For a *given* trial, we can apply our standard analysis tools and metrics**
  - **Fourier transform, mean and power calculations, etc…**
- **When trying to analyze the *ensemble* (i.e. *all* trials) of possible outcomes, we find ourselves in need of *new* tools and metrics**

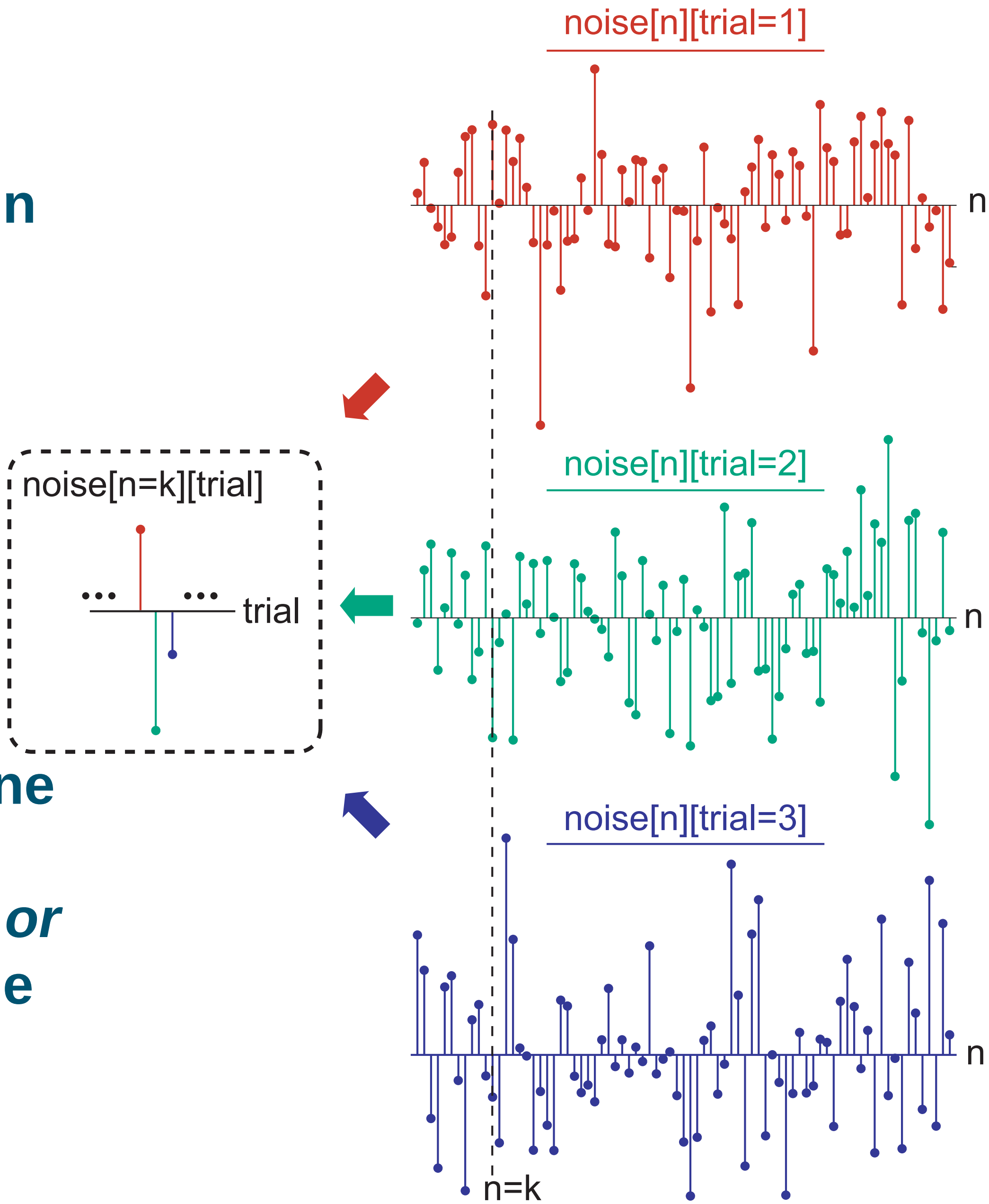


Tools and Metrics for Random Processes

- **Assume that random processes we will deal with have the properties of being *stationary* and *ergodic***
  - **True for noise in many practical systems**
  - **Greatly simplifies analysis**
- **Examine in both time and frequency domains**
  - **Time domain**
    - Introduce the concept of a *probability density function* (PDF) to characterize behavior of signals at a given sample time
    - Use PDF to calculate mean and variance
      - Similar to mean and power of non-random signals
  - **Frequency domain**
    - We will discuss a more proper framework in the next lecture
    - For now, we will simply use Fourier analysis (i.e., Fast Fourier Transform, *FFT*) on signals from *individual* trials

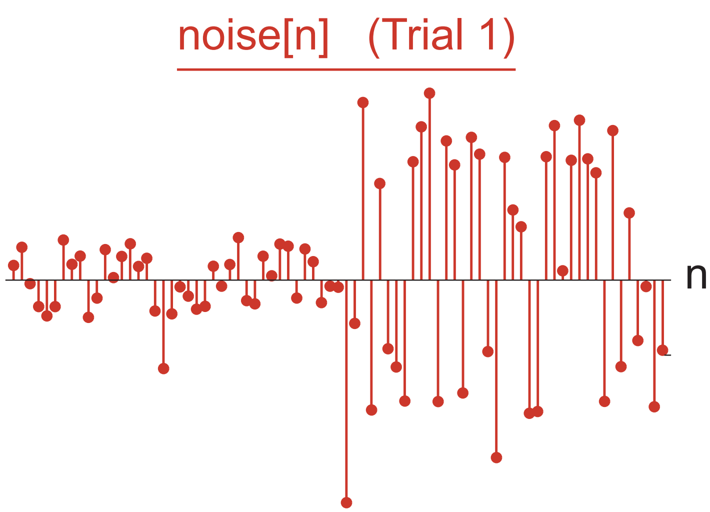
Stationary and Ergodic Random Processes

- **Stationary**
  - **Statistical behavior is independent of *shifts* in *time* in a given trial:**
    - Implies *noise[k]* is statistically indistinguishable from *noise[k+N]*
- **Ergodic**
  - **Statistical *sampling* can be performed at one sample time (i.e., *n=k*) across *different* trials, *or* across different sample times of the *same* trial with no change in the statistical result**

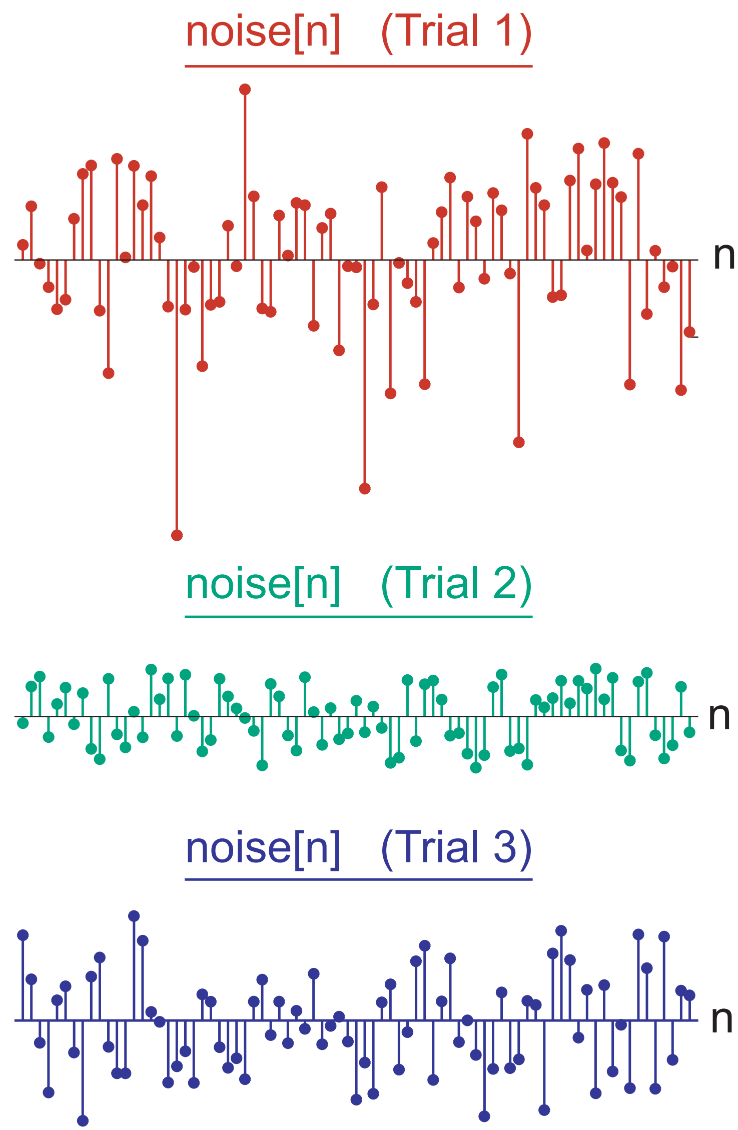


Examples

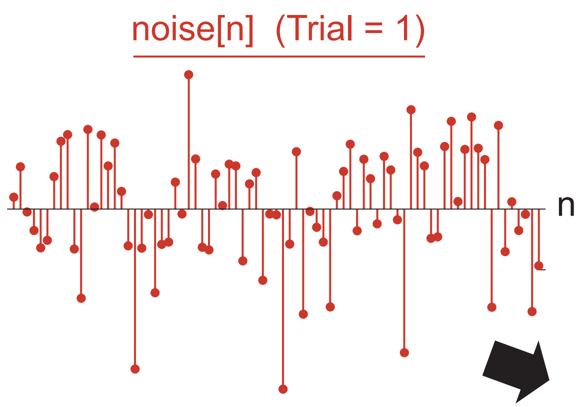
- **Non-Stationary**

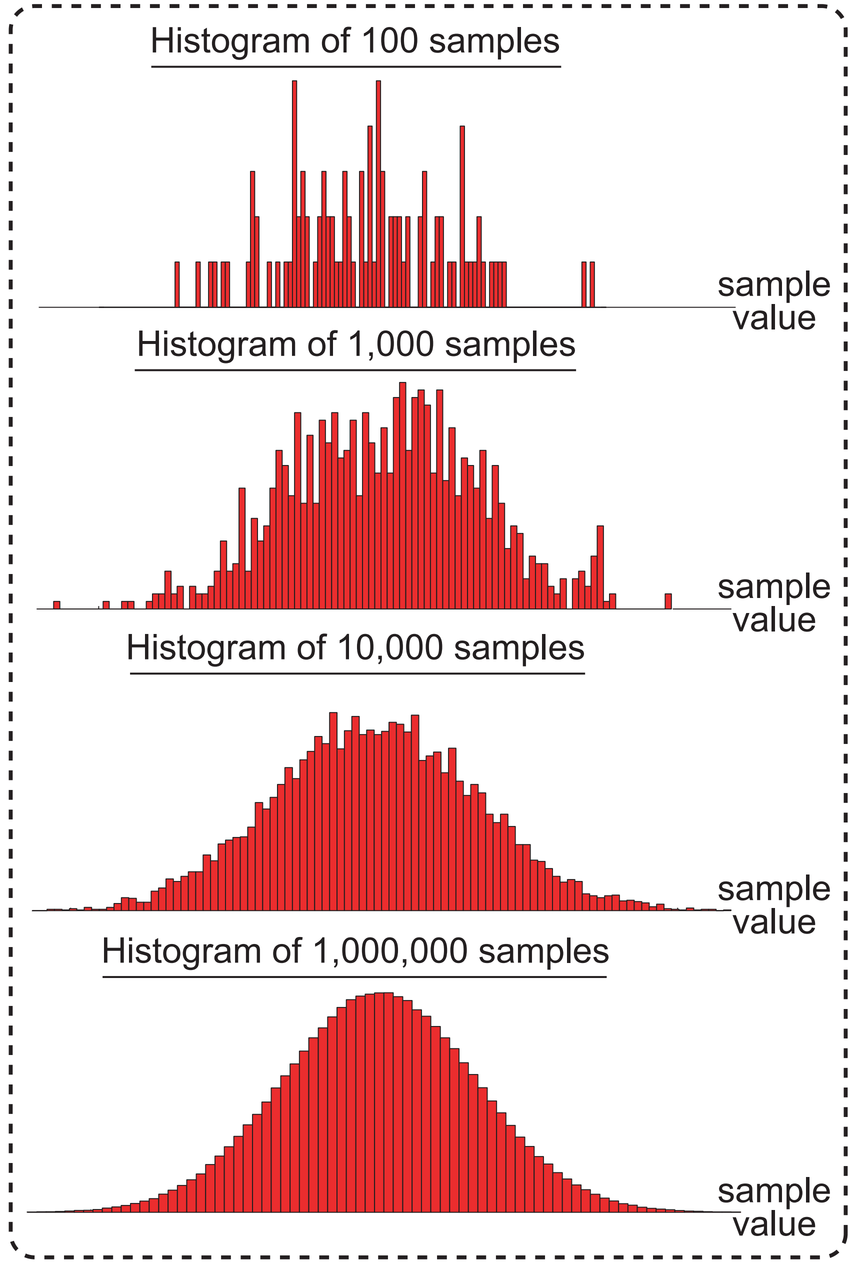


- **Stationary, but Non-Ergodic**



Experiment to see Statistical Distribution



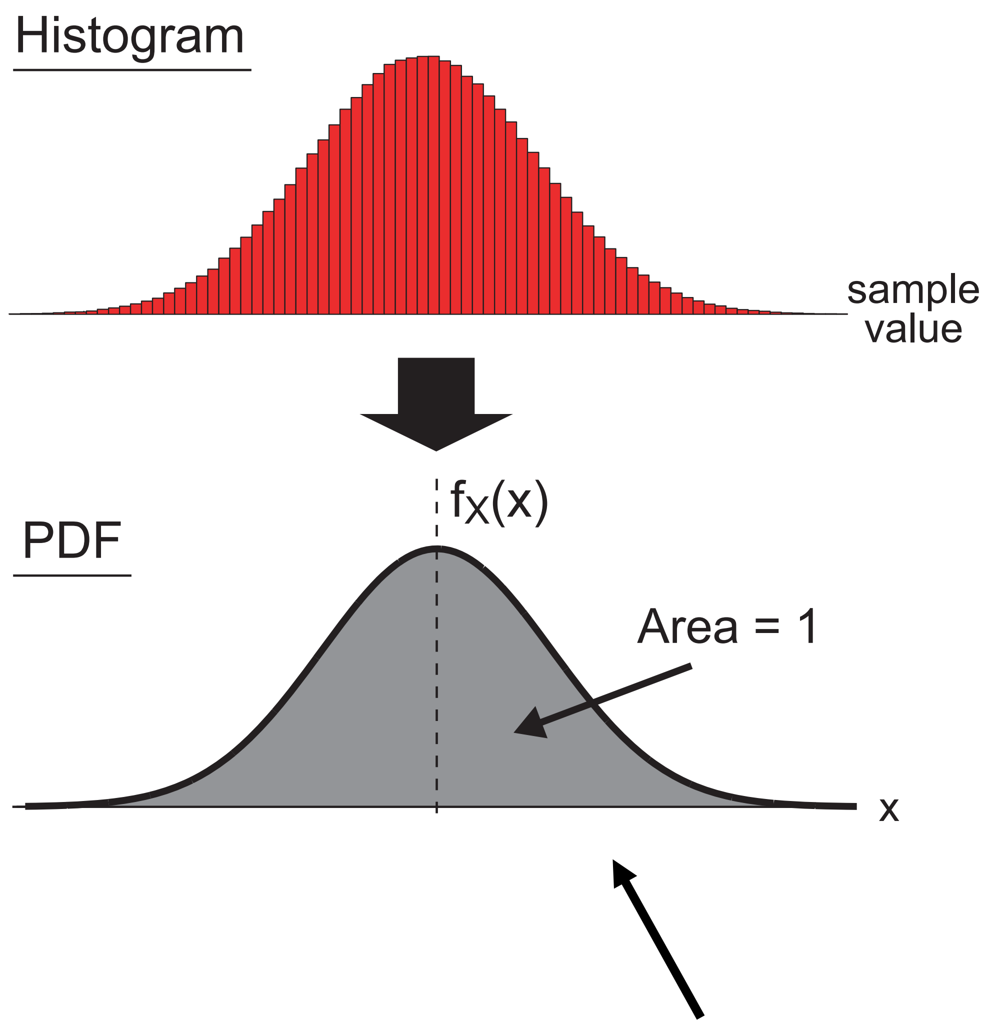


- **Create histograms of sample values from trials of increasing lengths**
- **Assumption of stationarity implies histogram should converge to a shape known as a probability density function (PDF)**

Formalizing the PDF Concept

- **Define *x* as a random variable whose PDF has the same shape as the histogram we just obtained**
- **Denote PDF of *x* as $f_X(x)$**
  - **Scale $f_X(x)$ such that its overall area is 1**

$$\Rightarrow \int_{-\infty}^{\infty} f_X(x) = 1$$

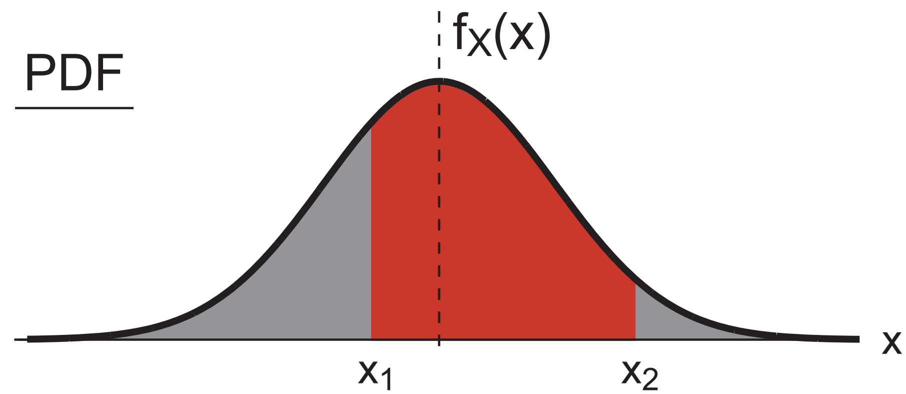


This shape is referred to as a *Gaussian* PDF

Formalizing Probability

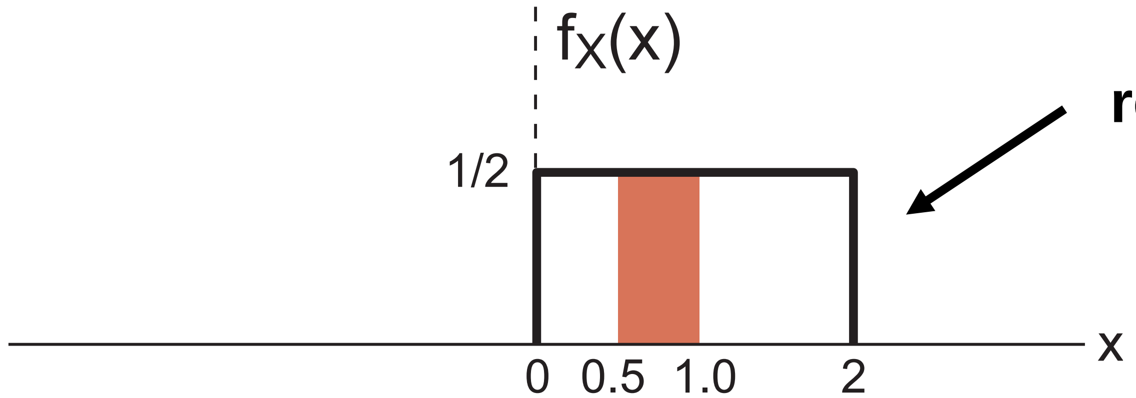
- The *probability* that random variable *x* takes on a value in the range of x_1 to x_2 is calculated from the PDF of *x* as:

$$\text{Prob}(x_1 \leq x \leq x_2) = \int_{x_1}^{x_2} f_X(x)dx$$



  - Note that probability values are always in the range of 0 to 1
  - Higher probability values imply greater likelihood that the event will occur

Example Probability Calculation



This shape is referred to as a *uniform* PDF

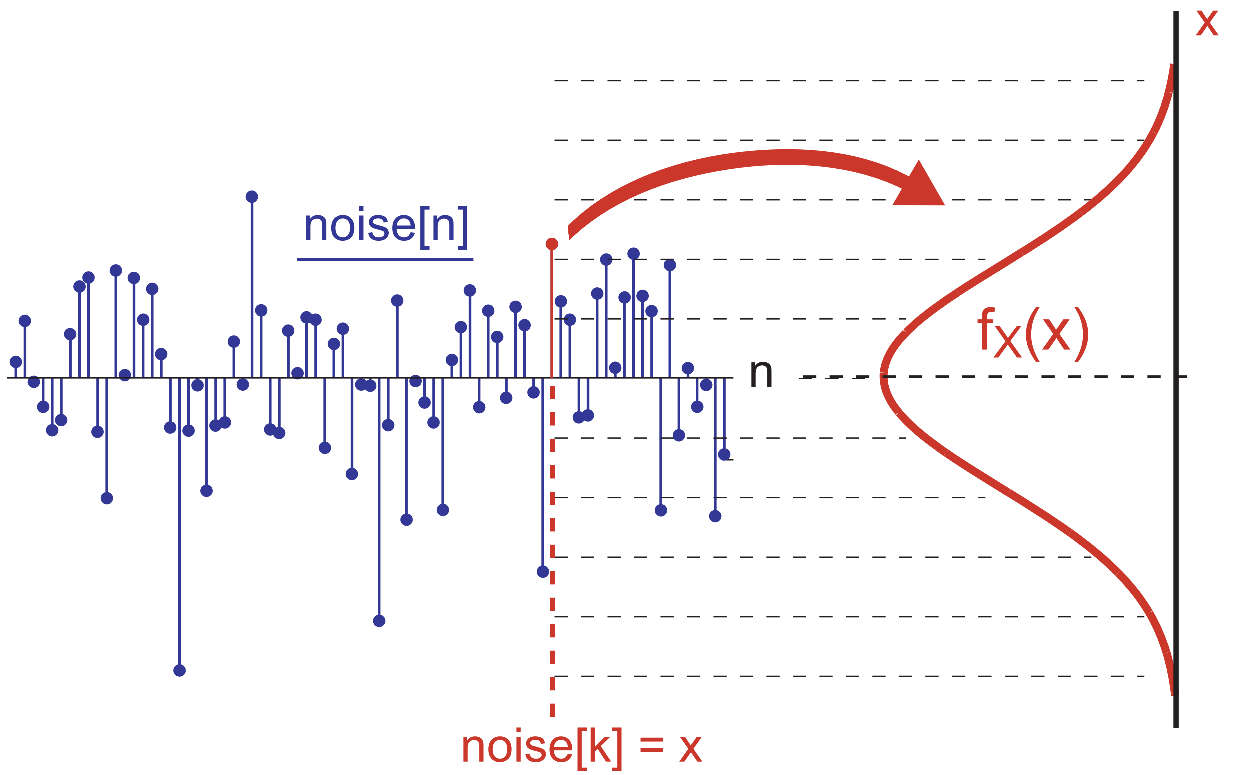
- **Verify that overall area is 1:**

$$\int_{-\infty}^{\infty} f_X(x)dx = \int_0^2 0.5\,dx \;= \boxed{1}$$

- **Probability that x takes on a value between 0.5 and 1.0:**

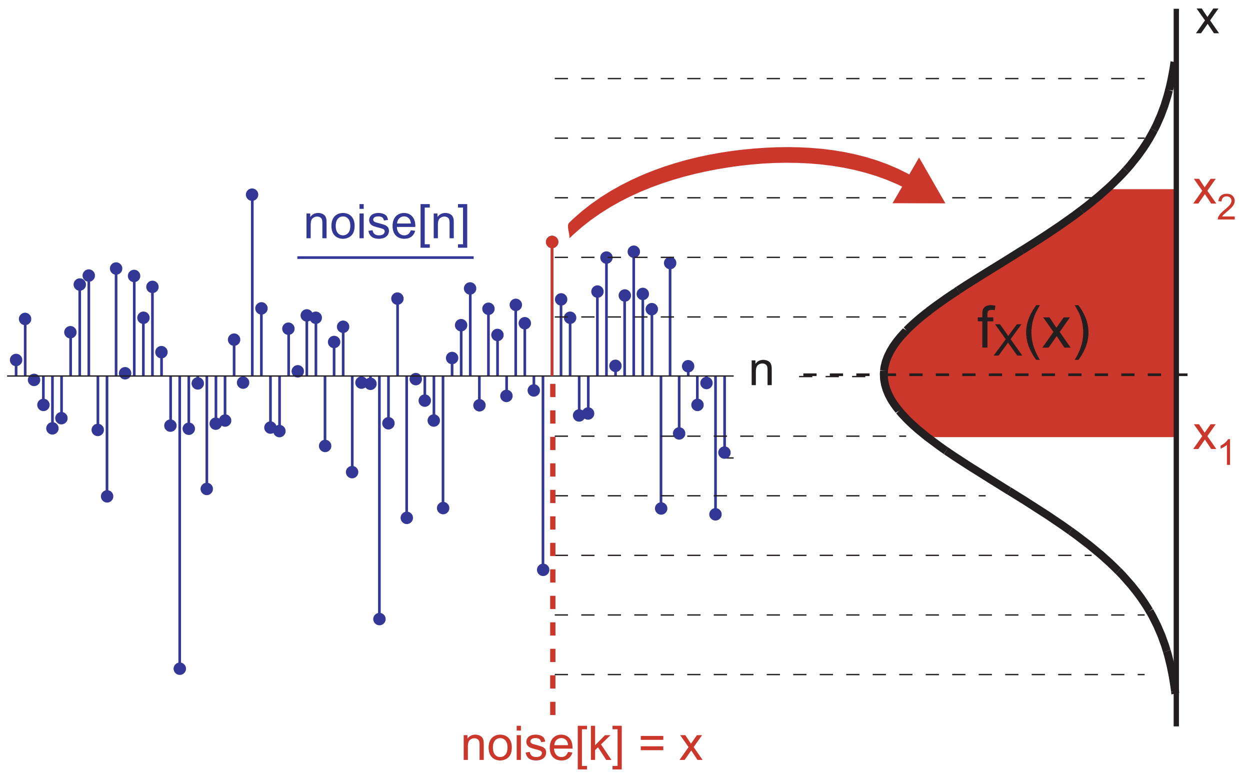
$$\mathrm{Prob}(0.5 \leq x \leq 1.0) = \int_{0.5}^{1.0} 0.5\,dx \;= \boxed{0.25}$$

Examination of Sample Value Distribution



- **Assumption of ergodicity implies the value occurring at a *given* time sample, *noise[k]*, across *many different* trials has the *same PDF* as estimated in our previous experiment of *many* time samples and *one* trial**
- **We can model *noise[k]* as the random variable *x***

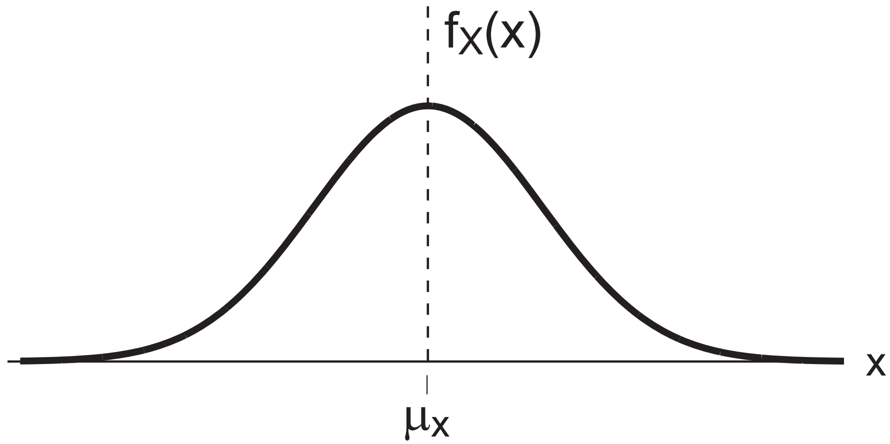
Probability Calculation



- **In a given trial, the *probability* that *noise[k]* takes on a value in the range of x_1 to x_2 is computed as**

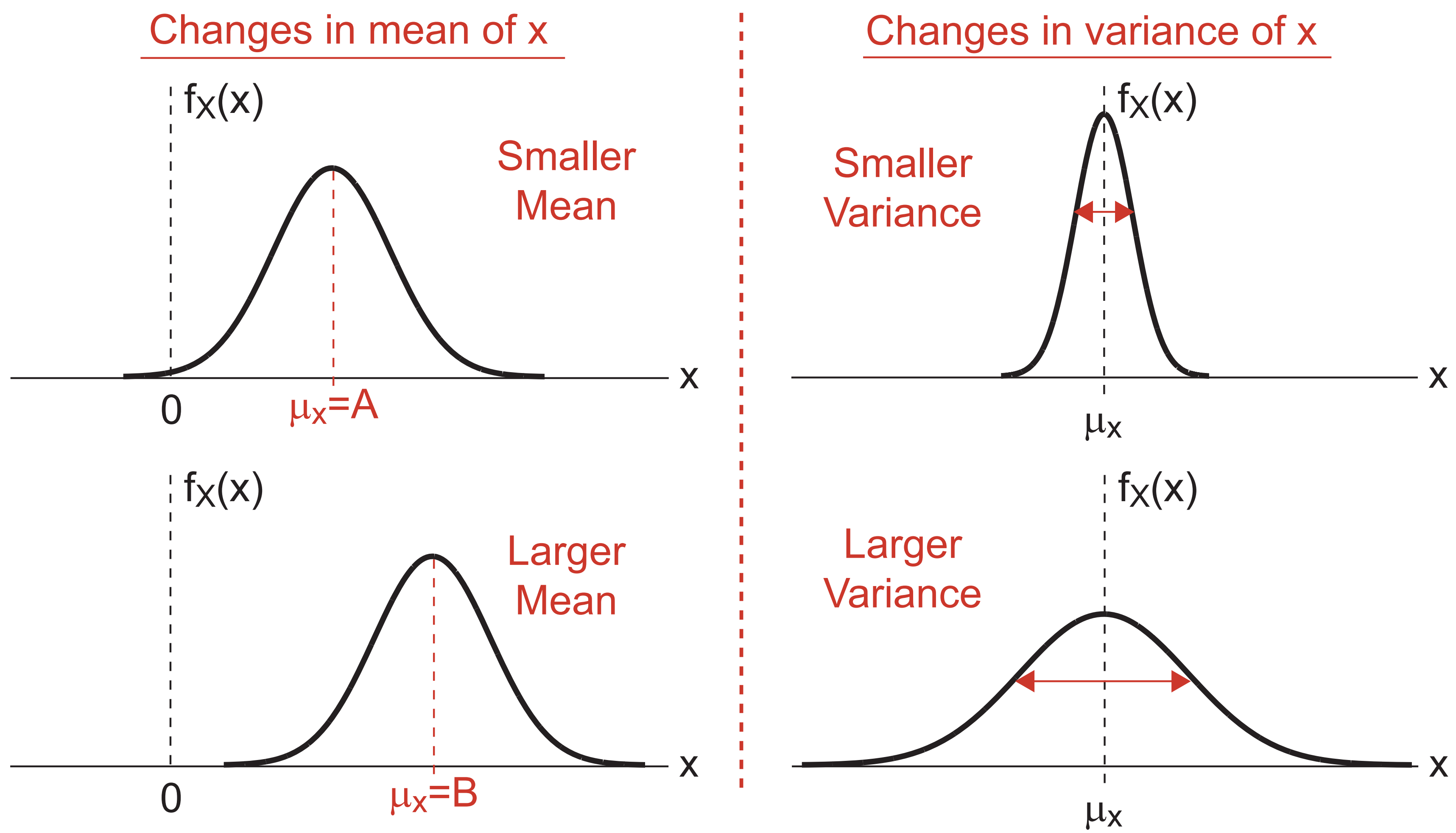
$$\mathrm{Prob}(x_1 \leq x \leq x_2) = \int_{x_1}^{x_2} f_X(x)dx$$

Mean and Variance



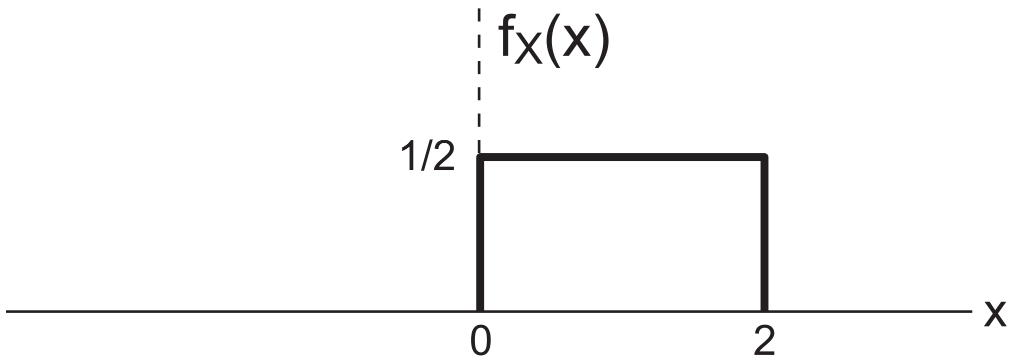
- The mean of random variable *x*, μ_x, corresponds to its average value
  - Computed as $$\mu_x = \int_{-\infty}^{\infty} x f_X(x) dx$$
- The variance of random variable *x*, σ_x^2, gives an indication of its variability
  - Computed as $$\sigma_x^2 = \int_{-\infty}^{\infty} (x - \mu_x)^2 f_X(x) dx$$
  - Similar to power of a signal

Visualizing Mean and Variance from a PDF



- **Changes in mean shift the *center of mass* of PDF**
- **Changes in variance narrow or broaden the PDF**
  - **Note that area of PDF must always remain equal to one**

Example Mean and Variance Calculation



- **Mean:**

$$\mu_x = \int_{-\infty}^{\infty} x f_X(x)dx = \int_0^2 x\frac{1}{2}dx = \frac{1}{4}x^2\Big|_0^2 = \boxed{1}$$

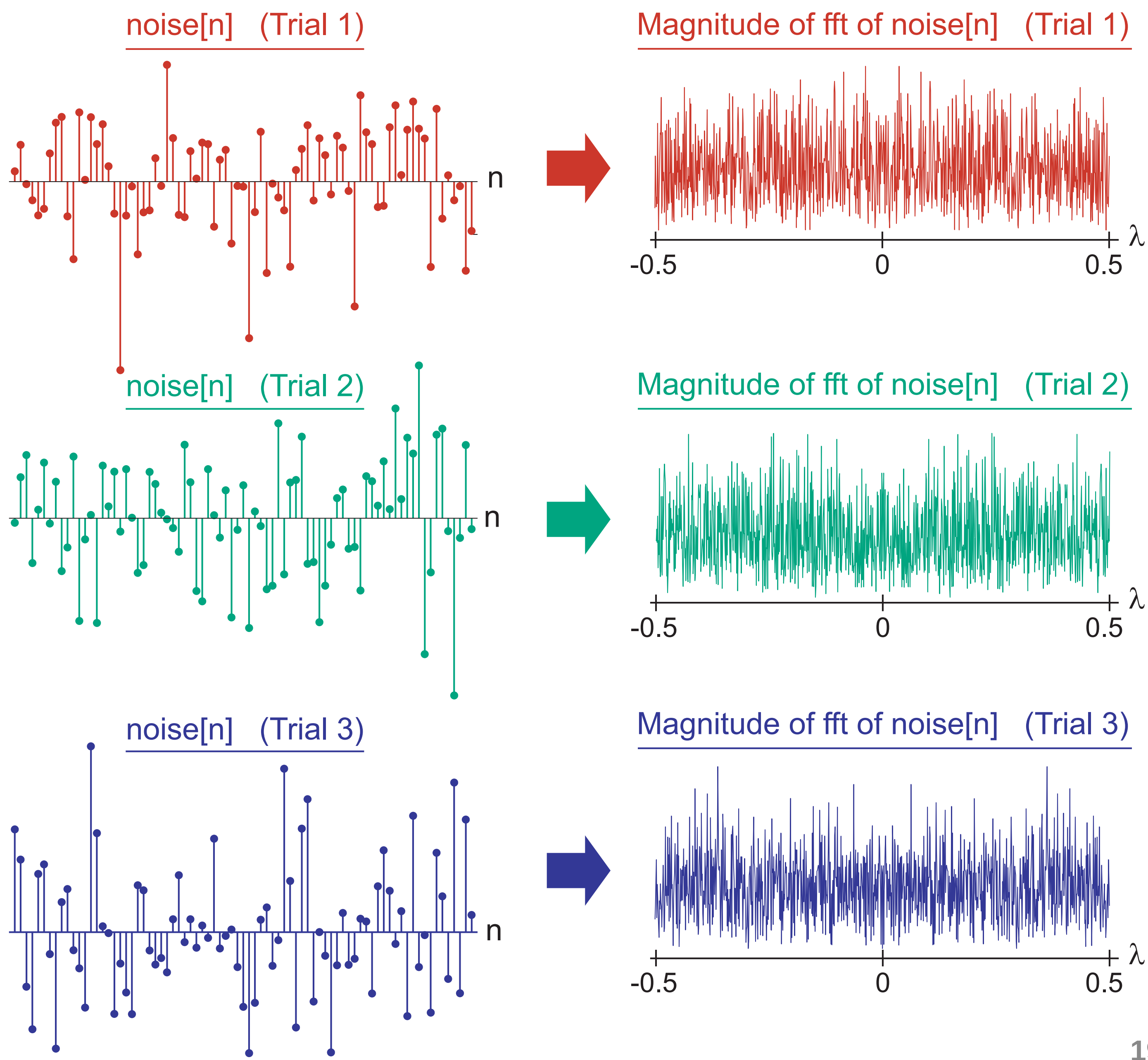
- **Variance:**

$$\sigma_x^2 = \int_{-\infty}^{\infty} (x-\mu_x)^2 f_X(x)dx = \int_0^2 (x-1)^2\frac{1}{2}dx$$

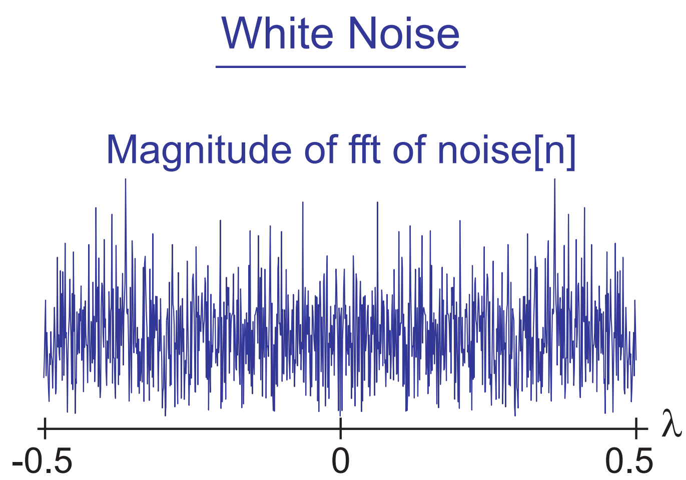
$$= \frac{1}{6}(x-1)^3\Big|_0^2 = \frac{1}{6}+\frac{1}{6} = \boxed{\frac{1}{3}}$$

Frequency Domain View of Random Process

- **It is valid to take the *FFT* of a sequence from a given trial**
- **However, notice that the *FFT* result changes across trials**
  - **Fourier Transform of a random process is undefined !**
  - **We need a new tool called spectral analysis**

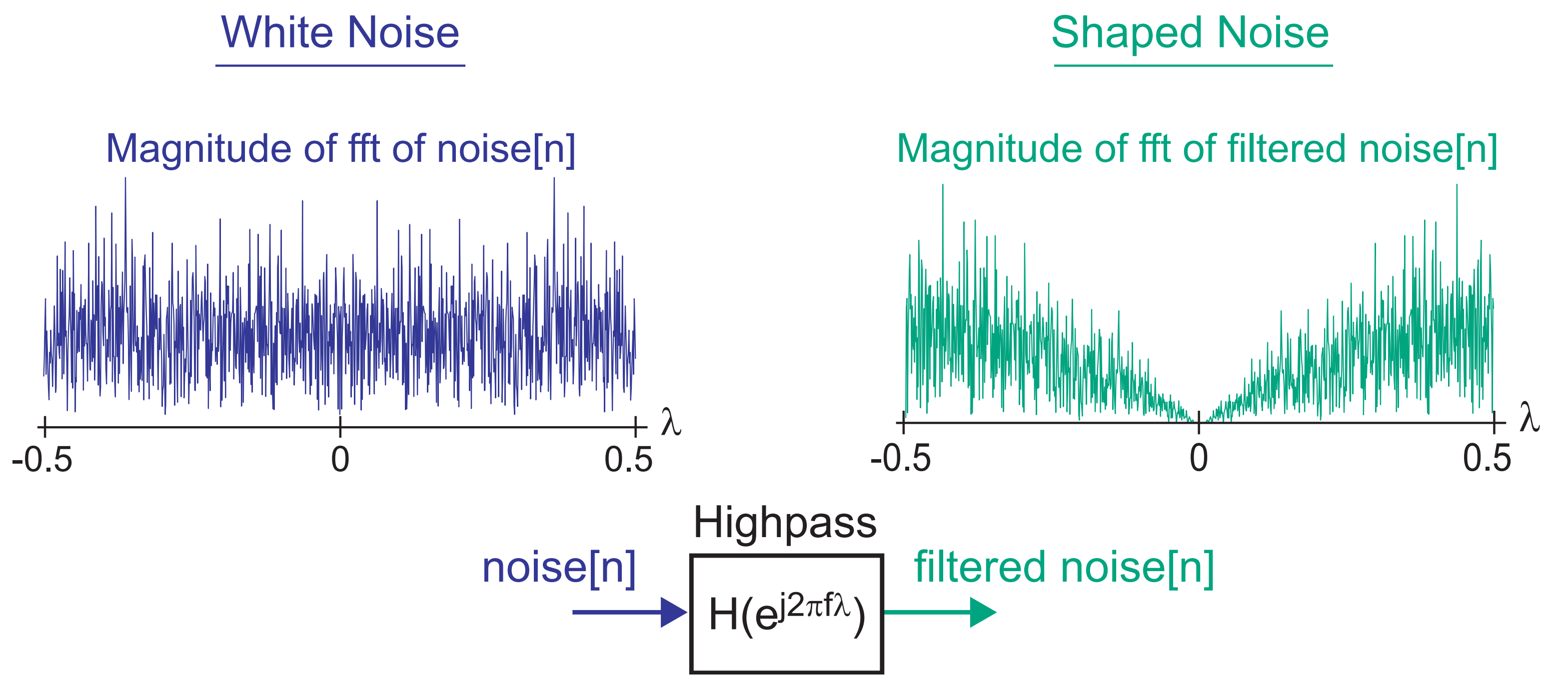


White Noise



- **When the *FFT* result looks relatively flat, we refer to the random process as being white**
  - **Note: this type of noise source is often used for calibration of advanced stereo systems**

Shaped Noise



- **Shaped noise occurs when white noise is sent into a filter**
  - ***FFT* of shaped noise will have frequency content according to the type of filter**
    - Example: highpass filter yields shaped noise with only high frequency content

Summary

- Discrete-time processes provide a useful context for studying the properties of noise
  - Analog circuits often convert real world (continuous-time) signals into discrete-time signals
- Signal-to-noise ratio is a key metric when examining the impact of noise on a system
- Noise is best characterized by using tools provided by the study of random processes
  - We will assume all noise processes we deal with are stationary and ergodic
  - Key metrics are mean and variance
  - Frequency analysis using direct application of Fourier Transforms is fine for one trial, but not valid when considering the ensemble of a random process

We will consider spectral analysis for continuous-time signals in the next lecture